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HOMEOMORPHISMS WITH VARIOUS MEASURE SHADOWING

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Abstract. In this paper we introduce two notions of measure shadowing for homeomorphsims, and study the relationship between them.

1. Introduction

Measure theory is one of the most useful tools for studying dynamical systems. In particular using measure theory enable us to study a property for a homeomorphism by using different measures, and to obtain results for that property regardless to measure theoretical point of view. For instance Morales in [1] showed that there is no expansive homeomorphsim from S^1 to S^1 by using notion of measure expansivity. Shadowing and expansivity are fundamental notions in theory of dynamical system. Here we introduce different defintions for measure shadowing and investigate equivalence relation among them.

Definition 1.1. Let (X, d) be a metric space, $\delta > 0$, and $f : X \to X$ a homeomorphism. A subsequence $\{x_i\}_{i\in\mathbb{Z}} \subset X$ is called δ-pseudo orbit if

$$d(f(x_i), x_{i+1}) < \delta$$

for all $i \in \mathbb{Z}$.

Definition 1.2. Let (X, d) be a metric space, $\varepsilon > 0$, and $f : X \to X$ a homeomorphism. We say that a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset X$ is ε shadowed by a real orbit if there is $x \in X$ such that

$$d(f^i(x), x_i) \leq \varepsilon \tag{1.1}$$

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for all $i \in \mathbb{Z}$.

To define the ε-shadowing in usual way the inequality (1.1) should be strict, but one can easily show that these two definitions are equivalent. Next we are going to define the product space by using the definition provided in [2, p5].

Definition 1.3. For $i \in \mathbb{Z}$ let $(X_i, \mathcal{B}_i, \mu_i)$ be a probability space. Let

$$Y = \prod_{i \in \mathbb{Z}} X_i,$$

and $\pi_j : \prod_{i \in \mathbb{Z}} X \to X_j$ be the corresponding projection map on j-th component. Let $i_1 < ... < i_k$ be a finite subsequence of $\mathbb{Z}$, and A_{i_j} be an arbitary element of $\mathcal{B}_{i_j}$ for some $j \in \{1, ..., k\}$. Then

$$\bigcap_{j=1}^{j=k} \pi_{i_j}^{-1}(A_{i_j})$$

is called a measurable rectangle. We denote the σ-algebra generated by such subsets of Y by $\mathcal{B}$. Define $\rho : \mathcal{B} \to \mathbb{R}^+$ by giving the above rectangle the value $\prod_{j=1}^{k} \mu_{i_j}(A_{i_j})$, then ρ extends to a probability measure μ on $(Y, \mathcal{B})$ (for more detail see [2, Theorem 0.2 and 0.4]). The probability measure $(X, \mathcal{B}, \mu)$ is called the product space of the spaces $(X_i, \mathcal{B}_i, \mu_i)$.

Hereafter we assume that (X, d) is a compact metric space, and $f : X \to X$ is a homeomorphism. Moreover we denote

$$\Phi_f(\delta) = \{(x_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} X \mid (x_i)_{i \in \mathbb{Z}} \text{ is a } \delta\text{-pseudo orbit}\},$$

and

$$S_f(\varepsilon, \delta) = \{(x_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} X \mid (x_i)_{i \in \mathbb{Z}} \in \Phi_f(\delta) \text{ is } \varepsilon \text{ shadowed by some point in X}\}.$$

Remark 1.4. Note that $\Phi_f(\delta)$ and $S_f(\varepsilon, \delta)$ are Borel sets.

Proof. Let $\pi_j : \prod_{i \in \mathbb{Z}} X \to X$ be the corresponding projection map on i-th component. If

$$A = \bigcup_{x \in X} \{x\} \times B_\delta(f(x)),$$

and

$$A_i = (\pi_i \times \pi_{i+1})^{-1}(A),$$

then

$$\Phi_f(\delta) = \bigcap_{i\in\mathbb{Z}} A_i.$$

We show that A is an open set and this implies that A_i's are Borel sets and consequently $\Phi_f(\delta)$ is Borel set. Let $(y,z)\in A$, then by definition of A we have $d(f(y),z) < \delta$. Because f is continuous there is $\delta' > 0$ such that if $d(x,y) < \delta'$, then $d(f(x),f(y)) < \frac{\delta - d(z,f(y))}{2}$. Now one can easily show that

$$(y,z) \in B_{\delta'}(y) \times B_{\frac{\delta + d(z,f(y))}{2}}(f(y)) \subset A,$$

and so A is open set.
To prove that $S_f(\varepsilon,\delta)$ is Borel set we show that

$$\bigcup_{x\in X}\prod_{i\in\mathbb{Z}}\overline{B_\varepsilon(f^i(x))},$$

is Borel set. Then because

$$S_f(\varepsilon,\delta) = \left(\bigcup_{x\in X}\prod_{i\in\mathbb{Z}}\overline{B_\varepsilon(f^i(x))}\right)\cap \Phi_f(\delta),$$

$S_f(\varepsilon,\delta)$ is also borel set. We show that

$$\bigcup_{x\in X}\prod_{i\in\mathbb{Z}}\overline{B_\varepsilon(f^i(x))} = \bigcap_{n\in\mathbb{N}}\bigcup_{x\in X}\left(\bigcap_{|i|\le n}\pi_i^{-1}(\overline{B_\varepsilon(f^i(x))})\right). \tag{1.2}$$

Let $(x_i)_{i\in\mathbb{Z}} \in \bigcap_{n\in\mathbb{N}}\bigcup_{x\in X}\left(\bigcap_{|i|\le n}\pi_i^{-1}(\overline{B_\varepsilon(f^i(x))})\right)$, then for any $n\in\mathbb{N}$ there is $z_n\in X$ such that

$$d(x_i, f^i(z_n)) \le \varepsilon,$$

for all $-n \le i \le n$. Since X is compact we may assume that $\{z_n\}_{n=1}^\infty$ is convergent to $z\in X$. This implies that if $j\in\mathbb{Z}$ is large enough then

$$d(x_i, f^i(z)) \le d(x_i, f^i(z_j)) + d(f^i(z_j), f^i(z)) \le \varepsilon + d(f^i(z_j), f^i(z)),$$

and since $d(f^i(z_j), f^i(z)) \to 0$ as $j\to\infty$, we conclude that

$$d(x_i, f^i(z)) \le \varepsilon.$$

So $(x_i)_{i\in\mathbb{Z}} \in \bigcup_{x\in X}\prod_{i\in\mathbb{Z}}\overline{B_\varepsilon(f^i(x))}$. On the other hand

$$(x_i)_{i\in\mathbb{Z}} \in \bigcup_{x\in X}\prod_{i\in\mathbb{Z}}\overline{B_\varepsilon(f^i(x))},$$

implies that

$$(x_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \overline{B_\varepsilon(f^i(x))},$$

for some $x \in X$ and so $(x_i)_{i\in\mathbb{Z}} \in \bigcap_{|i|\le n} \pi_i^{-1}(\overline{B_\varepsilon(f^i(x))})$ for all $n \in \mathbb{N}$. This implies (1.2), and proof is compeleted. $\square$

DEFINITION 1.5. We say that f satisfies product measure shadowing if for all $\varepsilon > 0$ there exists a $\varepsilon > \delta > 0$ such that

$$\mu^{\mathbb{Z}}(\Phi_f(\delta) - S_f(\varepsilon, \delta)) = 0$$

for all $\mu \in M^*(X)$. Here $M^*(X)$ is the set of all non-atomic Borel probability measures on X, and $\mu^{\mathbb{Z}}$ is the product measure generated by μ.

Note that if f satisfies product measure shadowing, then the set of δ-pseudo orbit which are not shadowed has a measure zero for all nonatomic product measures.

LEMMA 1.6. *Let (X, d) be a compact metric space. Then the identity map on X satisfies product measure shadowing.*

Proof. Let $\varepsilon > 0$, and let $0 < \delta < \varepsilon$ be arbitrary real numbers. Notice that $\Phi_{id}(\delta)$ and $S_{id}(\varepsilon, \delta)$ are left shift invariant, and if $A \subset \prod_{i\in\mathbb{Z}} X$ is measurable and invariant under left shift map then $\mu^{\mathbb{Z}}(A) = 0$ or 1 for all $\mu \in M^*(X)$. If $\mu^{\mathbb{Z}}(\Phi_{id}(\delta)) = 0$ then since $S_{id}(\varepsilon, \delta) \subset \Phi_{id}(\delta)$ we have nothing to prove. Otherwise $\mu^{\mathbb{Z}}(\Phi_{id}(\delta)) = 1$, and this implies $\mu^{\mathbb{Z}}(\Phi_{id}(\delta)^c) = 0$. In this case we show that there is $z_0 \in X$ such that $supp(\mu) \subset B_\delta(z_0)$. Suppose by contradiction that there are $x, y \in supp(\mu)$ such that $d(x, y) > 2\delta$. Then there are U_x and U_y neighbourhoods of x and y such that $\mu(U_x) > 0$, $\mu(U_y) > 0$, and $d(U_x, U_y) > \delta$. If

$$(x_i)_{i\in\mathbb{Z}} \in ... \times X \times U_x \times U_y \times X \times .....,$$

then

$$d(x_0, x_1) > d(U_x, U_y) > \delta,$$

and

$$d(id(x_0), (x_1)) = d(x_0, x_1).$$

So $(x_i)_{i\in\mathbb{Z}}$ can not be a δ- pseudo orbit, and we conclude that $... \times X \times U_x \times U_y \times X \times .....$ is subset of $\Phi_{id}(\delta)^c$. But this implies

$$0 = \mu^{\mathbb{Z}}(\Phi_{id}(\delta)^c) \ge \mu^{\mathbb{Z}}(... \times X \times U_x \times U_y \times X \times .....) = \mu(U_x)\mu(U_y) > 0,$$

which is contradiction and $supp(\mu) \subset B_\delta(z_0)$ for some $z_0 \in X$.
Let $(x_i)_{i\in\mathbb{Z}} \in S_{id}(\varepsilon, \delta)$ then there is $z \in X$ such that $\{x_i\}_{i\in\mathbb{Z}} \subset \overline{B_\varepsilon(z)}$

which implies $(x_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \overline{B_\varepsilon(z)}$. On the other hand if $(x_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \overline{B_\varepsilon(z)} \cap \Phi_{id}(\delta)$ for some $z \in X$ then $\{x_i\}_{i\in\mathbb{Z}} \subset \overline{B_\varepsilon(z)}$, and consequently $(x_i)_{i\in\mathbb{Z}} \in S_{id}(\varepsilon,\delta)$. So we have

$$S_{id}(\varepsilon,\delta) = (\bigcup_{z\in X} \prod_{i\in\mathbb{Z}} \overline{B_\varepsilon(z)}) \cap \Phi_{id}(\delta),$$

and

$$S_{id}(\varepsilon,\delta)^c = (\bigcap_{z\in X} (\prod_{i\in\mathbb{Z}} \overline{B_\varepsilon(z)})^c) \cup \Phi_{id}(\delta)^c.$$

But $\mu^{\mathbb{Z}}(\Phi_{id}(\delta)^c) = 0$, moreover $supp(\mu) \subset \overline{B_\delta(z_0)}$ so we have

$$\mu^{\mathbb{Z}}((\prod_{i\in\mathbb{Z}} \overline{B_\varepsilon(z_0)})^c) \le \mu^{\mathbb{Z}}(\bigcup_{i\in\mathbb{Z}} \pi_i^{-1}(\overline{B_\delta(z_0)}^{\,c}) = 0.$$

Whence $\mu^{\mathbb{Z}}(S_{id}(\varepsilon,\delta)) = 1$ and the proof is completed. □

DEFINITION 1.7. We say that f satisfies measure shadowing if for all $\varepsilon > 0$ there exists a $\varepsilon > \delta > 0$ such that

$$\mu(\Phi_f(\delta) - S_f(\varepsilon,\delta)) = 0$$

for all $\mu \in M^*(X^{\mathbb{Z}})$. Here $M^*(X^{\mathbb{Z}})$ is the set of all nonatomic Borel measures on the infinite product set $X^{\mathbb{Z}}$.

The notion of measure shadowing is stronger than that of product measure shadowing. In fact, if μ is a nonatomic probability measure on X then $\mu^{\mathbb{Z}}$ is also a nonatomic probability measure on $X^{\mathbb{Z}}$. We show that the identity map on the unit circle S^1 does not satisfy measure shadowing.

LEMMA 1.8. *The identity map on S^1 does not satisfy measure shadowing.*

Proof. Suppose that $\frac{1}{6} > \varepsilon > \delta > 0$. There are $x_0, ...x_k \in S^1$ such that

$$d(x_i, x_{i+1}) < \frac{\delta}{2} \text{ for all } i = 0, ..., k-1, \tag{1.3}$$

and

$$max\{d(x_i, x_j) \mid i, j = 0, ..., k\} > 3\varepsilon. \tag{1.4}$$

For $i = 0, ..., k$ define a nonatomic measure $\mu_i \in M^*(S^1)$ as following

$$\mu_i(E) = \frac{1}{Leb B_{\frac{\delta}{2}}(x_i)} Leb(E \cap B_{\frac{\delta}{2}}(x_i)),$$

where $E \subset S^1$ is a Borel set and Leb is the Lebegue measure on S^1. Then

$$\mu = \ldots \times Leb \times \ldots \times Leb \times \mu_0 \times \mu_1 \times \ldots \times \mu_k \times Leb \times Leb \times \ldots.$$

is a nonatomic Borel probability measure on ${S^1}^{\mathbb{Z}}$. We show that

$$\ldots \times B_{\frac{\delta}{2}}(x_0) \times \ldots \times B_{\frac{\delta}{2}}(x_0) \times B_{\frac{\delta}{4}}(x_0) \times B_{\frac{\delta}{4}}(x_1) \times \ldots \times B_{\frac{\delta}{4}}(x_k) \times B_{\frac{\delta}{2}}(x_0) \times B_{\frac{\delta}{2}}(x_0) \times \ldots.,$$

is subset of $\Phi_{id}(\delta) - S_{id}(\varepsilon, \delta)$, and since

$$\mu(\ldots \times B_{\frac{\delta}{2}}(x_0) \times \ldots \times B_{\frac{\delta}{2}}(x_0) \times B_{\frac{\delta}{4}}(x_0) \times B_{\frac{\delta}{4}}(x_1) \times \ldots \\ \times B_{\frac{\delta}{4}}(x_k) \times B_{\frac{\delta}{2}}(x_0) \times B_{\frac{\delta}{2}}(x_0) \times \ldots.) > 0,$$

the proof will be completed. Let

$$(y_i)_{i \in \mathbb{Z}} \in \ldots \times B_{\frac{\delta}{2}}(x_0) \times \ldots \times B_{\frac{\delta}{2}}(x_0) \times B_{\frac{\delta}{4}}(x_0) \times B_{\frac{\delta}{4}}(x_1) \times \ldots \\ \times B_{\frac{\delta}{4}}(x_k) \times B_{\frac{\delta}{2}}(x_0) \times B_{\frac{\delta}{2}}(x_0) \times \ldots.,$$

then

$$d(y_i, y_{i+1}) < d(y_i, x_0) + d(x_0, y_{i+1}) < \delta \ (i < 0 \text{ or } i > k),$$

and

$$d(y_i, y_{i+1}) < d(y_i, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, y_{i+1}) < \delta \ (0 \leq i \leq k),$$

hence $(y_i)_{i \in \mathbb{Z}} \in \Phi_{id}(\delta)$. If $(y_i)_{i \in \mathbb{Z}} \in S_{id}(\varepsilon, \delta)$ then there is $z_0 \in s^1$ such that

$$d(z_0, y_i) \leq \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

But (1.4) implies that there are $0 \leq i, j \leq k$ such that $d(x_i, x_j) > 3\varepsilon$. So

$$d(y_i, y_j) \leq d(y_i, z_0) + d(y_j, z_0) \leq 2\varepsilon,$$

and

$$3\varepsilon < d(x_i, x_j) \leq d(y_i, y_j) + d(y_i, x_i) + d(y_j, x_j) < 2\varepsilon + \frac{\delta}{2} + \frac{\delta}{2} \leq 3\varepsilon,$$

which is contradiction. □

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